Lecture 13 (Jan 4)

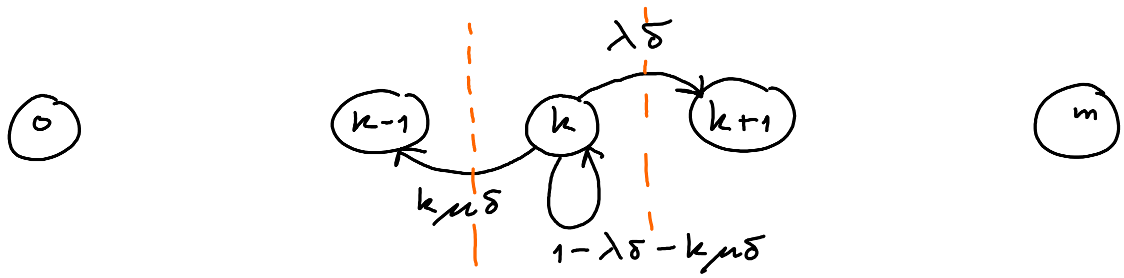
Erlang B via Markov chain and discrete-time approximation

Call initiation : Poisson process with rate λ
request

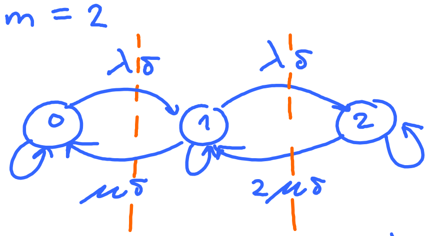
$\lambda \uparrow$ aggregated rate (total)

combination of rates from all users.

Assume : Infinite users.



Example $m = 2$



$p_0 \times \lambda \cancel{\delta} = p_1 \mu \cancel{\delta}$

$p_1 = p_0 \dfrac{\lambda}{\mu} = A p_0$

$p_1 \lambda \cancel{\delta} = p_2 \, 2\mu \cancel{\delta}$

$p_2 = \dfrac{\lambda}{2\mu} p_1 = \dfrac{1}{2} A \, p_1$

$= \dfrac{1}{2} A \times A p_0 = \dfrac{1}{2} A^2 p_0$

$p_0 + p_1 + p_2 = 1$

$p_0 + A p_0 + \frac{1}{2} A^2 p_0 = 1 \Rightarrow p_0 = \dfrac{1}{1 + A + \frac{1}{2}A^2}$

$p_1 = A p_0 = \dfrac{A}{1 + A + \frac{1}{2}A^2}$

$p_2 = \dfrac{\frac{1}{2}A^2}{1 + A + \frac{1}{2}A^2}$

A^m

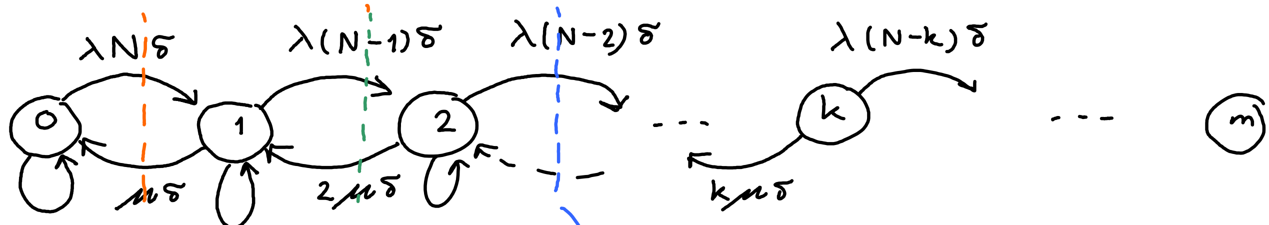
In general, m channels $\Rightarrow$ $P_m = \dfrac{\frac{A^m}{m!}}{\sum_{k=0}^{m} \frac{A^k}{k!}}$

Engset Model
(Engsett)

- Finite ~~#~~ users : N users.
- Each user generate new call request with rate λ

  (might be better to call it λ_u)

- m channels
- call duration $\sim \mathcal{E}(\mu)$ $\left(\text{average} = \frac{1}{\mu}\right)$



$$P_0 \times \lambda N \cancel{\delta} = P_1 \times \mu \cancel{\delta}$$

$$P_1 = P_0 \frac{\lambda N}{\mu} = P_0 A N$$

$$P_1 \times \lambda(N-1)\cancel{\delta} = P_2 \times 2\mu\cancel{\delta}$$

$$P_2 = A \frac{(N-1)}{2} P_1$$

$$= A \frac{(N-1)}{2} N A P_0$$

$$= \frac{N(N-1)}{2} A^2 P_0$$

$$= \binom{N}{2} A^2 P_0$$

$$P_2 \times \lambda(N-2)\cancel{\delta} = P_3 \times 3\mu\cancel{\delta}$$

$$P_3 = \frac{\lambda}{\mu} \frac{N-2}{3} \times P_2$$

$$= A \frac{N-2}{3} \times \frac{N(N-1)}{2} A^2 P_0$$

$$= A^3 \frac{N(N-1)(N-2)}{3 \times 2 \times 1} P_0 = A^3 \binom{N}{3} P_0$$

$\vdots$

$\vdots$

$$p_k = A^k \binom{N}{k} p_0 \qquad 0 \le k \le m$$

$$\sum_{k=0}^{m} p_k = 1 \quad \Rightarrow \quad \sum_{k=0}^{m} A^k \binom{N}{k} p_0 = 1$$

$$p_0 = \frac{1}{\sum_{k=0}^{m} \binom{N}{k} A^k}$$

Normalization factor $\rightarrow Z(m, N)$

$$p_k = A^k \binom{N}{k} p_0 = \frac{A^k \binom{N}{k}}{Z(m,N)}$$

$$p_m = \frac{A^m \binom{N}{m}}{Z(m,N)}$$ ← time congestion probability.

~~call blocking probability~~

To find call blocking probability,

consider s slots (s is very large)

about $p_k \times s$ slots will be in state k.

Each of this will have a new call request with probability

$$(N-k)\lambda\delta$$

$$\#\text{ new calls} = (N-k)\lambda\delta \times (p_k \times s)$$

$$\text{Total } \#\text{ new calls} = \sum_{k=0}^{m} (N-k)\lambda\delta\, p_k\, s$$

$$\#\text{ blocked call} = (N-m)\lambda\delta \times (p_m \times s)$$

$$\text{Blocked call proportion (probability)} = \frac{(N-m)\cancel{\lambda\delta}\, p_m \times \cancel{s}}{\sum^{m} (N-k)\cancel{\lambda\delta}\, p_k \times \cancel{s}}$$

(probability) $\sum_{k=0}^{m} (N-k) \cancel{\lambda s}\, p_k \times \cancel{s}$

$$= \frac{(N-m)\, p_m}{\sum_{k=0}^{m} (N-k)\, p_k}$$

call congestion

For Erlang B, $= \frac{p_m}{\sum_{k=0}^{m} p_k} = p_m$